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NONCONVEX OPTIMIZATION APPROACH TO COST VOLUME PROFIT ANALYSIS

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ABSTRACT

This paper intends to develop a new cost-volume-profit (CVP) analysis based on optimization and sphere packing theories [1, 2]. The original cost-volume-profit (CVP) model was first introduced by Hess and Mann in 1903 and generalized later to multiproduct case. It seems a little attention has been paid to the extension of existing models of CVP analysis when its parameters such as sales, prices, and costs vary simultaneously over a given period. For this purpose, we propose a new approach to profitability analysis based on a notion of set of profitability conditions with respect to CVP parameters which is nonconvex. The main difficulty for handling CVP analysis is nonconvexity of the set of profitability conditions. To overcome this, we apply the penalty function method in order to find a feasible point in the nonconvex set of the set of profitability conditions. Finding a feasible point allows to construct other subsets of the set of profitability conditions based on sphere packing theory. The

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approach also provides practical suggestions and recommendations for managers to choose a set of optimal CVP parameters. The proposed approach is illustrated on some examples providing numerical results.

KEYWORDS
management decison making, CVP analysis, profitability analysis, optimization, penalty method, sphere packing theory

1. Introduction

Cost Volume Profit (CVP) analysis is a management tool that is used to planning and making decisions in the business environment. This plays important role in predicting the impact of changes in costs, sales, and volume on a business's future profits. According to Horngren and Foster (2010), the basic CVP model is considered under the following assumptions and limiting conditions: costs and revenues are linear, selling prices are constant, prices of production inputs are constant, all costs can be categorized into their fixed and variable elements, total fixed costs remain constant, total variable costs are proportional to volume. Fixed costs are constant over the range of the analysis while variable costs are proportional to volume. CVP is used to determine the level of sales which is required to recover all costs incurred during the period. Since the objective of CVP analysis is to determine the level of sales which is required to achieve a targeted level of profit, break even analysis may be considered as a special case of CVP. If sales fall below the break even point, losses will be incurred. Management must determine the break even point to compute the margin of safety, which indicates how much sales may decrease from the targeted level before the company will incur losses. The objective of CVP analysis is to determine the volume of sales to achieve a targeted level of profit. There are many works devoted to break even and CVP analysis, but most of them deal with one type of product case. G.O. Nworie[22] and Abdullahi et al.[3] examine CVP analysis as a decision making tool in small businesses. They suggest that small businesses be exposed to CVP analysis and other management accounting techniques to increase efficiency. Enyi [6] examines the multiproduct CVP analysis applications based on the Weighted Contribution Margin. In [8] CVP analysis has been applied in all businesses and any industry, whether large or small. Revenue analysis of mobile rice milling business has been considered in [10]. Application of CVP analysis for the development of stingless bees farming in Malaysia has been shown in [12]. The research done in [14] recommended that block industries should use CVP analysis as a main tool in profit planning because of its impact, efficiency, and accuracy in the rationalization of profit maximization objective. To improve the efficiency of reusable products and obtain an environmental benefit, CVP analysis has been done in [13]. CVP analysis for milk production of selected EU countries was done in [9, 15]. In [16] CVP analysis for the battery production has been analyzed. Ihemeje, Okereafor, and Ogungbangbe (2015) noted that managerial incompetence is one of the major problems faced by firms when using management accounting models such as CVP. Recently, new applications of sphere packing theory [2] have been applied to CVP analysis [24] which considers a set of profitability analysis with respect to its parameters. In this work only convex properties of the set of profitability conditions have been considered. But when all parameters of CVP analysis vary simultaneously, the set of profitability conditions is not convex. For this reason direct application of sphere packing theory to CVP analysis may fail. Taking into account the above existing literature and noting that less attention has been paid to define the sets of CVP parameters such as sales, prices, and variable costs which guarantees the profitability of industries, we propose a new mathematical methodology based on nonconvex optimization method with combination of sphere packing theory. Optimization method allows us to a find a feasible point from the nonconvex set of profitability analysis while the sphere packing deals with this point to generate other sets of profitability.

In this paper, we define and find sets of profitability conditions with respect to volume, price, and cost based on penalty function method and sphere packing theory [2]. The proposed approach was illustrated on some examples.

The paper is organized as follows. Penalty method for defining points from a set of profitability conditions has been defined in Chapter 2. Application of sphere packing theory to profitability analysis of a company has been given in Chapter 3. The numerical implementation of the proposed approach has been illustrated in Chapter 4.

2. Profit analysis with varying parameters

The total profit of a company for a multi-product case can be written as [19, 24]:

$$\pi = \sum_{j=1}^{n} p_j x_j - \sum_{j=1}^{n} c_j x_j - F \tag{1}$$

where, π-total profit, p_j-price per unit of j-th product, x_j-quantity of product sold, c_j-variable cost per unit of j-th product, F-total fixed cost, $p_j > c_j$.

It is important for a company to operate profitable, in other words, company's profit must be nonnegative:

$$\pi \geq 0$$

or the set K defined by

$$K(p,x,c,F) = \{(p,x,c,F) \in R^{3n+1} | \sum_{j=1}^{n} (p_j - c_j)x_j \geq F,\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$p_j^{min} \leq p_j \leq p_j^{max},\ c_j^{min} \leq c_j \leq c_j^{max},\ F^{min} \leq F \leq F^{max}, p_j > c_j,\ j = 1, \dots, n\}$$

is nonempty.

The set $K(p,x,c,F)$ is called a set of profitability conditions. Clearly, the set K is nonconvex. It is important to find a feasible point in the set K in order to generate sets of profitability conditions with respect to its parameters p, c, and F. In order to find a feasible point of K, we apply the penalty method [25]. Let a point u be an arbitrary point in $\mathbb{R}^{3n+1}$.

Consider the problem

$$\min_{z \in D} f(z) = \|z - u\|^2, \ \ z = (p,x,c,F) \tag{2}$$

subject to:

$$g(z) = F - \sum_{j=1}^{n}(p_j - c_j)x_j \leq 0, \tag{3}$$

where

$$D = \{z \in R^{3n+1} |\ x_j^{min} \leq x_j \leq x_j^{max},\ p_j^{min} \leq p_j \leq p_j^{max},$$

$$c_j^{min} \leq c_j \leq c_j^{max},\ F^{min} \leq F \leq F^{max},\ p_j > c_j,\ j = 1, \dots, n\}.$$

It can be easily checked that if the following condition

$$F \leq \sum_{j=1}^{n}(p_j^{min} - c_j^{max})x_j^{min}$$

holds then the set K is nonemtpy.

Problem (2)-(3) is a nonconvex optimization problem.

Algorithm of Penalty Function Method(PFM)[25]

To solve the problem, we apply the penalty function approach. The advantage of the penalty method is to reduce the constrained optimization problem into an unconstrained one so that it can be easily solved by a standard optmization methods and algorithms.

Introduce the penalty function $p(z)$ as follows:

$$p(z) = (\max(g(z), 0))^2$$

and

$$L(z, C_k) = f(z) + C_k p(z)$$

Note that

$$\min_{z\in D} L(z, C_k) > -\infty,$$

here $\{C_k\}_{k=1}^{\infty}$ is a positive and monotonically increasing sequence. Algorithm of penalty method is implemented in the following steps[25]:

Step 1: Parameters C_1, m, eps are given and set $k = 1, C_1 = 10$, $m \in N$

Step 2: Solve the problem

$$\min_{z\in D} L(z, C_k).$$

Letz^k be a solution to the problem, that is $L(z^k, C_k) = \min_{z\in D} L(z^k, C_k)$.

Step 3: Set $u = z^k, \quad k = k + 1, \quad C_k = mC_{k-1}$

Step 4: Solve the problem

$$\min_{z\in D} L(z, C_k) = L(z^k, C_k).$$

Step 5: If $\|u - z^k\| \leq eps$ then algorithm stops and solution z^k is a local minimum or a stationary point. Otherwise, go to Step 3.

It is well known that the sequence $\{z^k, k = 1, 2, \dots\}$ generated by the algorithm of the penalty method converges to a solution of problem (2)-(3)[25].

We can generate as many feasible points of the set of profitability conditions as we choose different initial points in solving problem (2)-(3).

3. Profit analysis with a given unit variable cost

If a variable cost per unit of j-th product c_j is given, that is $c_j = \tilde{c_j}, j = 1, \dots, n$, then a set of profitability conditions is written as follows:

$$K_1(p, x, F) = \{(p, x, F) \in R^{2n+1} | \sum_{j=1}^{n} (p_j - \tilde{c}_j)x_j \geq F,\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$p_j^{min} \le p_j \le p_j^{max},\ F^{min} \le F \le F^{max},\ p_j > c_j,\ j = 1, \dots, n.\}$$

The set $K_1(p, x, F)$ is called a set of profitability conditions with respect to parameters (p, x, F). The set $K_1(p, x, F)$ is nonconvex. Let a point v be an arbitrary point in $\mathbb{R}^{2n+1}$. In order to find a feasible point in the set $K_1(p, x, F)$, we solve the following optimization problem.

$$\min_{z \in D_1} f_1(z) = \|z - v\|^2,\ \ z = (p, x, F) \tag{4}$$

subject to:

$$g_1(z) = F - \sum_{j=1}^{n} (p_j - \tilde{c}_j) x_j \le 0, \tag{5}$$

where

$$D_1 = \{z \in R^{2n+1} |\ x_j^{min} \le x_j \le x_j^{max},\ p_j^{min} \le p_j \le p_j^{max},$$

$$F^{min} \le F \le F^{max},\ p_j > \tilde{c}_j,\ j = 1, \dots, n\}.$$

Using the penalty function algorithm, we find a feasible point $\tilde{z} = (\tilde{p}, \tilde{x}, \tilde{c}, \tilde{F})$.

4. Profit analysis with a given unit variable cost and fixed cost

Assume that a variable cost c and fixed cost F in (1) are given. Suppose that $c = \tilde{c}$ and $F = \tilde{F}$. Denote a set of profitability conditions with respect to price and volume by K_2:

$$K_2(p,x) = \{(p,x) \in R^{2n} | \sum_{j=1}^{n} (p_j - \tilde{c}_j)x_j \geq \tilde{F},\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$p_j^{min} \leq p_j \leq p_j^{max},\ p_j > \tilde{c}_j,\ j = 1, \dots, n.\}$$

It is clear that the set K_2 is not convex and problem (2)-(3) reduces to the following optimization problem:

$$\min_{z \in D_2} f_1(z) = \|z - u\|^2,\ \ z = (p,x) \tag{6}$$

subject to:

$$g_2(z) = \tilde{F} - \sum_{j=1}^{n} (p_j - \tilde{c}_j)\tilde{x}_j \leq 0, \tag{7}$$

where

$$D_2 = \{z \in R^{2n} | \ x_j^{min} \leq x_j \leq x_j^{max},\ p_j^{min} \leq p_j \leq p_j^{max},$$

$$p_j > \tilde{c}_j,\ j = 1, \dots, n\}.$$

Similarly, solving problem (6)- (7) by penalty function approach, we find a feasible point $\bar{z} = (\bar{p}, \bar{x})$ of the set $K_2(p,x)$.

5. Profit analysis with a given price

If a price is given as $p = \tilde{p}$ in (1), then we obtain a set of profitability conditions with respect to volume, variable cost and fixed cost:

$$K_3(x, c, F) = \{(x, c, F) \in R^{2n+1} | \sum_{j=1}^{n} (\tilde{p}_j - c_j)x_j \geq F,\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$F^{min} \leq F \leq F^{max},\ \tilde{p}_j > c_j,\ j = 1, \dots, n\}.$$

The set $K_3(x, c, F)$ is called a set of profitability conditions with respect to parameters (x, c, F). The set $K_3(x, c, F)$ is nonconvex. Let a point v be an arbitrary point in $\mathbb{R}^{2n+1}$. In order to find a feasible point in the set $K_3(x, c, F)$, we solve the following optimization problem.

$$\min_{z \in D_3} f_3(z) = \|z - v\|^2,\ \ z = (x, c, F) \tag{8}$$

subject to:

$$g_3(z) = F - \sum_{j=1}^{n} (\tilde{p}_j - c_j)x_j \leq 0, \tag{9}$$

where

$$D_3 = \{z \in R^{2n+1} |\ x_j^{min} \leq x_j \leq x_j^{max},\ c_j^{min} \leq c_j \leq c_j^{max},$$

$$F^{min} \leq F \leq F^{max},\ \tilde{p}_j > c_j,\ j = 1, \dots, n\}.$$

Using the penalty function algorithm, we find a feasible point $\tilde{q} = (\tilde{p}, \tilde{x}, \tilde{c}, \tilde{F})$ of the set of profitability conditions $K_3(x,c,F)$.

6. Profit analysis with a given price and fixed cost

Assume that price and fixed cost are given in (1). In other words, $p = \tilde{p}$ and $F = \tilde{F}$. The set of profitability conditions K_4 with respect to parameters (x, c) can be written as follows:

$$K_4(x,c) = \{(x,c) \in R^{2n} | \sum_{j=1}^{n} (\tilde{p}_j - c_j)x_j \geq \tilde{F},\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$c_j^{min} \leq c_j \leq c_j^{max}, \tilde{p}_j > c_j,\ j = 1, \ldots, n\}$$

Problem (2)-(3) reduces

$$\min_{z \in D_4} f_4(z) = \|z - u\|^2,\ \ z = (x,c) \tag{10}$$

subject to:

$$g_4(z) = \tilde{F} - \sum_{j=1}^{n} (\tilde{p}_j - c_j)x_j \leq 0, \tag{11}$$

where

$$D_4 = \{z \in R^{2n} |\ x_j^{min} \leq x_j \leq x_j^{max},$$

$$c_j^{min} \leq c_j \leq c_j^{max},\ \tilde{p}_j > c_j,\ j = 1, \ldots, n\}.$$

Similarly, if we solve problem (10)-(11) by the penalty method algorithm, we obtain a feasible point in the set of profitability conditions $K_4(x, c)$.

7. Application of sphere packing theory

The subsets of profitability conditions can be also generated using sphere packing theory which deals with packing non overlapping spheres of the maximum volume in a given set [2]. In order to examine profitability analysis from a view point of sphere packing theory it is worth mentioning the latest result on this theory [1].

Let $B(u^0, r)$ be a ball with a center $u^0 \in R^n$ and radius $r \in R$.

$$B(u^0, r) = \{x \in R^n | \|x - u^0\| \leq r\}, \tag{12}$$

here $\langle , \rangle$ denotes the scalar product of two vectors in R^n, and $\| \cdot \|$ is Euclidean norm.

Let D be a polytope given by the following linear inequalities.

$$D = \{x \in R^n | \langle a^i, x \rangle \leq b_i, i = \overline{1, m}\}, a^i \in R^n, b_i \in R.$$

Assume that D is a compact set which is not congruent to a sphere and $intD \neq \emptyset$. Clearly, D is a convex set in R^n.

Lemma 7.0.1. *[1]. $B(u^0, r) \subset D$ if and only if*

$$\langle a^i, u^0 \rangle + r\|a^i\| \leq b_i, i = \overline{1, m}. \tag{13}$$

$$r \geq 0. \tag{14}$$

Problem of inscribing a sphere with the maximum radius into D reduces to the following linear programming.

$$\max_{(x,r)} r \tag{15}$$

subject to

$$\langle a^i, x\rangle + r\|a^i\| \le b_i,\ i = \overline{1,m}, \tag{16}$$

$$r \ge 0. \tag{17}$$

Let $z^0 = (p^0, x^0, c^0, F^0)$ be a solution to problem (2)-(3). Let us consider the following cases.

Case 1. Now for given parameters of p^0, c^0 and F^0, we introduce the set of profitability conditions with respect to volume as follows[24] :

$$K_x = \left\{ x \in R^n \mid \sum_{j=1}^{n} (p_j^0 - c_j^0)x_j \ge F^0,\ x_j^{min} \le x_j \le x_j^{max},\ j = 1, \ldots, n \right\} \tag{18}$$

where, x_j^{min}, x_j^{max}-minimum and maximum capacity volumes for j-th product, $j = 1, \ldots, n$ and $\|x\|$ denotes the norm of the vector x in R^n. Denote by $B(u^0, r^0)$ a sphere with a center $x^0 \in R^n$ and radius $r^0 \in R$, $r^0 > 0$:

$$B(u^0, r^0) = \{x \in R^n \mid \|x - u^0\| \le r^0\}. \tag{19}$$

It is easy to see that any point $y \in B$ can be presented as

$$y = u^0 + \alpha \frac{h}{\|h\|} \tag{20}$$

for any $h \in R^n$ and $0 \le \alpha \le r^0$.

According to sphere packing theory[1], we must find a sphere $B(u^0, r^0)$ with the maximum radius r^0 such that $B(u^0, r^0) \subset K_x$. For this purpose, we rewrite problem (15)-(17), taking into account the constraints of K_x given by (18), as the following linear programming problem

$$\max\ r \tag{21}$$

$$\sum_{j=1}^{n}(p_j^0 - c_j^0)x_j - r\sqrt{\sum_{j=1}^{n}(p_j^0 - c_j^0)^2} \geq F^0 \tag{22}$$

$$-x_j + r \leq -x^{min}, j = 1, \ldots, n \tag{23}$$

$$x_j + r \leq x_j^{max}, j = 1, \ldots, n. \tag{24}$$

Let (u^0, r^0) be a solution to the above problem and if we take any

$$h = (h_1, h_2, \ldots, h_n) \in R^n$$

then

$$x^h = (x_1, \ldots, x_n) = \left(u_1^0 + r^0\frac{h_1}{\|h\|}, \ldots, u_i^0 + r^0\frac{h_i}{\|h\|}, \ldots, u_n^0 + r^0\frac{h_n}{\|h\|}\right)$$

$$\bar{x}^h = (\bar{x}_1, \ldots, \bar{x}_n) = \left(u_1^0 - r^0\frac{h_1}{\|h\|}, \ldots, u_i^0 - r^0\frac{h_i}{\|h\|}, \ldots, u_n^0 - r^0\frac{h_n}{\|h\|}\right)$$

$$\|h\| = \sqrt{h_1^2 + \ldots, h_n^2}$$

and $x^h, \bar{x}^h \in B(u^0, r^0)$, consequently $x^h, \bar{x}^h \in K_x$ satisfy profitability conditions (18).

It means that in order to ensure company's profit, the company must keep its volumes in the following intervals:

$$u_j^0 - r^0 \frac{h_j}{\|h\|} \leq x_j \leq u_j^0 + r^0 \frac{h_j}{\|h\|}, \ j = 1, \ldots, n. \tag{25}$$

Denote by Z_x the following set

$$Z_x = \left\{ x \in R^n \mid u_j^0 - r^0 \frac{h_j}{\|h\|} \leq x_j \leq u_j^0 + r^0 \frac{h_j}{\|h\|}, \ j = 1, \ldots, n \right\}.$$

It is clear that

$$Z_x \subset K_x. \tag{26}$$

Case 2. Now assume that volumes of products $x_j^0, j = 1, \ldots, n$, fixed and variable costs are given. The set of profitability conditions with respect to price is defined as follows [24]:

$$K_p = \left\{ p \in R^n \ \mid \sum_{j=1}^{n} (p_j - c_j^0) x_j^0 \geq F^0, \ p_j^{min} \leq p_j \leq p_j^{max}, \ j = 1, \ldots, n \right\}.$$

Define a sphere $\bar{B}(z^0, \bar{r}^0)$ with a center z^0 and a radius $\bar{r}^0$:

$$\bar{B}(z^0, \bar{r}^0) = \left\{ p \in R^n \mid \|p - z^0\| \leq \bar{r}^0 \right\}.$$

In order to check condition $\bar{B}(z^0, \bar{r}^0) \subset K_p$ with the maximum radius, according to sphere packing theory, we solve the following linear programming

$$r \to \max \tag{27}$$

$$\begin{aligned} \sum_{j=1}^{n} p_j x_j^0 - r\sqrt{\sum_{j=1}^{n}(x_j^0)^2} \geq \sum_{j=1}^{n} c_j^0 x_j^0 + F^0 \\ -p_j + r \leq -p_j^{min},\ j = 1, \dots, n \\ p_j + r \leq p_j^{max},\ j = 1, \dots, n. \end{aligned} \tag{28}$$

Let $p^* = (p_1^*, p_2^*, \dots, p_n^*)$ and r^* be a solution to the above problem then a point p constructed as

$$p = p^* + r^* \frac{h}{\|h\|} \in K_p$$

for any $h = (h_1, h_2, \dots, h_n) \in R^n$ belongs to K_p . Also, we can show that all points

$$p(\alpha) = p^* + \alpha \frac{h}{\|h\|} \in K_p$$

belong to K_p for $0 \leq \alpha \leq r^*$ and $h \in R^n$. If we take $h = (1, 1, \dots, 1) \in R^n$, then

$$p_j^* - r^* \frac{1}{\sqrt{n}} \leq p_j \leq p_j^* + r^* \frac{1}{\sqrt{n}},\ j = 1, \dots, n. \tag{29}$$

It means that in order to ensure company's profit, the company must keep its prices in the above intervals. Denote by Z_p the following set

$$Z_p = \left\{ p \in R^n \mid p_j^* - r^* \frac{1}{\sqrt{n}} \leq p_j \leq p_j^* + r^* \frac{1}{\sqrt{n}},\ j = 1, \dots, n. \right\}.$$

It is clear that

$$Z_p \subset K_p. \tag{30}$$

Case 3. Let the volumes x_j^0, prices p_j^0 of products and the fixed cost F^0 be given. Define the set of profitability conditions with respect to variable cost as follows[24]:

$$K_c = \left\{ c \in R^n \mid \sum_{j=1}^{n} (p_j^0 - c_j) x_j^0 \geq F^0,\ c_j^{min} \leq c_j \leq c_j^{max},\ j = 1, \ldots, n \right\}.$$

Define a sphere $B(v^0, r^0)$ with a center v^0 and a radius r^0:

$$B(v^0, r^0) = \left\{ c \in R^n \mid \|c - v^0\| \leq r^0 \right\}.$$

By analogy with formulas (37)-(38), in order to check condition $B(v^0, r^0) \subset K_c$ with the maximum radius, we solve the problem.

$$r \to \max \tag{31}$$

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j^0 + r \sqrt{\sum_{j=1}^{n} (x_j^0)^2} &\leq \sum_{j=1}^{n} p_j^0 x_j^0 - F^0 \\ -c_j + r &\leq -c_j^{min},\ j = 1, \ldots, n \\ c_j + r &\leq c_j^{max},\ j = 1, \ldots, n. \end{aligned} \tag{32}$$

Let $(c^*, r^*) = (c_1^*, c_2^*, \ldots, c_n^*, r^*)$ be a solution to problem (40)-(41). Then points c constructed as

$$c = c^* + r^* \frac{h}{\|h\|} \in K_c$$

for any $h = (h_1, h_2, \dots, h_n) \in R^n$ belong to K_c. Also, we can show that all points

$$c(\alpha) = c^* + \alpha \frac{h}{\|h\|} \in K_c$$

for $0 \le \alpha \le r^*$ and $h \in R^n$.

By analogy with case 3, it means that in order to ensure company's profit, the company must keep its variable costs in the intervals

$$c_j^* - r^* \frac{1}{\sqrt{n}} \le c_j \le c_j^* + r^* \frac{1}{\sqrt{n}}, \; j = 1, \dots, n. \tag{33}$$

Denote by Z_c the following set

$$Z_c = \left\{ c \in R^n \mid c_j^* - r^* \frac{1}{\sqrt{n}} \le c_j \le c_j^* + r^* \frac{1}{\sqrt{n}}, \; j = 1, \dots, n \right\}.$$

It is clear that

$$Z_c \subset K_c. \tag{34}$$

Finally, taking into account formulas (26), (30), and (34), we have

$$Z_x \bigcup Z_p \bigcup Z_c \subset K.$$

The sets Z_x, Z_p, and Z_c are regarded as approximation sets of the set of profitability conditions K.

8. Numerical Implementation

In order to illustrate the proposed approach numerically, we use the following given data of company.

$x_1^{min} = 8, x_1^{max} = 28, x_2^{min} = 5, x_2^{max} = 20, x_3^{min} = 2, x_3^{max} = 28,$
$c_1^{min} = 220, c_1^{max} = 320, c_2^{min} = 300, c_2^{max} = 530, c_3^{min} = 1000, c_3^{max} = 2500,\ p_1^{min} = 250, p_1^{max} = 800, p_2^{min} = 600, p_2^{max} = 1500, p_3^{min} = 3200, p_3^{max} = 5500,$
$F^{min} = 14000, F^{max} = 17000.$

For solving problem (2)-(3), an initial approximation point u^0 was chosen as

$$u^0 = (26, -7, 4, -250, 320, 2300, -400, 700, 6000, 15000).$$

Then problem (2)-(3) is written for the above data in the following:

$$\min_{z \in D} f(z) = (x_1 - 26)^2 + (x_2 + 7)^2 + (x_3 - 4)^2 + (c_1 + 250)^2 + (c_2 - 320)^2 + \\ +(c_3 - 2300)^2 + +(p_1 + 400)^2 + (p_2 - 700)^2 + (p_3 - 6000)^2 + (F - 15000)^2$$

subject to:

$$g(z) = F - \sum_{j=1}^{n} (p_j - c_j) x_j \leq 0$$

$$8 \leq x_1 \leq 28$$

$$5 \leq x_2 \leq 20$$

$$2 \leq x_3 \leq 7$$

$$220 \leq c_1 \leq 320$$

$$300 \leq c_2 \leq 530$$

$$1000 \leq c_3 \leq 2500$$

$$250 \leq p_1 \leq 800$$

$$600 \leq p_2 \leq 1500$$

$$3200 \leq x_3 \leq 5500$$

$$14000 \leq F \leq 17000.$$

Note this optimization problem belongs to a class of nonconvex optimization and has many local extremum points. Application of the Lagrange method to the problem does not guarantee always finding a global solution. Even finding a local solution by the Lagrange method has difficulties because of complementarity conditions greated by a box constraints imposed on parameters. That is why penalty function method is most suitable which finds at least one feasible point in a nonconvex set of profitability conditions.

The algorithm of penalty function method for solving the optimization problem coded in Matlab found the following solution:

$$z^0 = (p^0, x^0, c^0, F^0),$$

where $p^0 = (250, 700, 5500)$, $x^0 = (26, 5, 3.83)$, $c^0 = (220, 320, 2300)$, $F^{=}1500$.

Now fixing some parameters in the solution, we can generate other sets of profitability conditions based sphere packing theory. **Example for case 1**.

Table 1. Parameters

i	unit price, p^0	unit variable cost, c^0	min.level of x_i	max.level of x_i
1	250	220	8	28
2	700	320	5	20
3	5,500.0	2300	2	7

For example, consider a company with the fixed cost of $F^0 = 15,000.0$, price p^0 and variable cost c^0. These parameters are given in Table 1. Using these parameters, we find an optimal range of volumes to ensure company's profitability by solving problem (23)-(26) which is formulated as

$$\begin{aligned} r &\to \max, \\ 30x_1 + 380x_2 + 3200x_3 - 3222.62r &\geq 15000 \\ x_1 - r &\geq 8 \\ x_1 + r &\leq 28 \\ x_2 - r &\geq 5 \\ x_2 + r &\leq 20 \\ x_3 - r &\geq 2 \\ x_3 + r &\leq 7 \\ r &\geq 0. \end{aligned} \tag{35}$$

The problem has been solved numerically by Matlab. The optimal radius is $r^* = 2.31$. Solutions, also low and upper bounds of optimal ranges computed by formula (27) have been given in the following table.

Table 2. Solution for case 1

i	solution, x_i	low bound of x_i	upper bound of x_i
1	25.68	24.34	27
2	17.68	16.34	19
3	4.68	3.34	6.02

We can also represent the set of profitability conditions with respect to volume by the following system of inequalities:

$$\begin{aligned} 24.34 \le x_1 \le 27 \\ 16.34 \le x_2 \le 19 \\ 3.34 \le x_3 \le 6.02. \end{aligned} \tag{36}$$

Example for case 2.

Table 3. Parameters

i	volume of x^0	unit variable cost, c^0	min.level of p_i	max.level of p_i
1	26	220	250	800
2	5	320	600	1500
3	3.83	2,300.0	3200	5,500.0

Consider a company with the fixed cost of $F^0 = 15,000.0$ and fixed parameters of x^0 and c^0 given in Table 3. Problem (37)-(38) which corresponds to case 2 is formulated as follows

$$\begin{aligned} r \to \max, \\ 26p_1 + 5p_2 + 3.83p_3 - 26.74r \ge 31129.19 \\ p_1 - r \ge 250 \\ p_1 + r \le 800 \\ p_2 - r \ge 600 \\ p_2 + r \le 1500 \\ p_3 - r \ge 3200 \\ p_3 + r \le 5500 \\ r \ge 0. \end{aligned} \tag{37}$$

The problem has been solved numerically by Matlab. The optimal radius is $r^* = 275$. Solutions, also low and upper bounds of optimal ranges computed by formula

(39) have been given in the following table.

Table 4. Solution for case 2

i	solution, p_i	low bound of p_i	upper bound of p_i
1	525	366.22	683.77
2	1225	1066.22	1383.77
3	4884.97	4726.2	5043.74

Then the set of profitability conditions with respect to price is given by the following system of inequalities

$$366.22 \leq p_1 \leq 683.77$$

$$1066.22 \leq p_2 \leq 1383.77$$

$$4726.2 \leq p_3 \leq 5043.74.$$

Example for case 3.

Table 5. Parameters

i	volume of x^0	price, p^0	min.level of c_i	max.level of c_i
1	26	250	220.0	320
2	5	700	300	530.0
3	3.83	5500	1000	2500

Consider a company with the fixed cost of $F^0 = 15,000.0$ and fixed parameters of x^0 and p^0 given in Table 5. Problem (37)-(38) which corresponds to case 3 is formulated as follows:

In order to find a set of the profitability conditions with respect to variable cost, we solve problem (40)-(41) using parameters in Table 5 which is formulated as

$$
\begin{aligned}
r &\to \max \\
26c_1 + 5c_2 + 3.83c_3 + 26.74.47r &\leq 16066.98 \\
c_1 - r &\geq 220 \\
c_1 + r &\leq 320 \\
c_2 - r &\geq 300 \\
c_2 + r &\leq 530 \\
c_3 - r &\geq 1000 \\
c_3 + r &\leq 2500 \\
r &\geq 0.
\end{aligned}
\tag{38}
$$

The problem has been solved numerically by Matlab. The optimal radius is $r^* = 50$. Solutions, also low and upper bounds of optimal ranges computed by formula (42) have been given in the following table.

Table 6. Solution for case 3

i	solution, c_i	low bound of c_i	upper bound of c_i
1	270	241.13	298.86
2	350	321.13	378.86
3	1050	1021.13	1078.86

Then the set of profitability conditions with respect to price is given by the following system of inequalities

$$241.13 \leq c_1 \leq 298.86$$

$$321.13 \leq c_2 \leq 378.86$$

$$1021.13 \leq c_3 \leq 1078.86.$$

9. Conclusion

In this paper, the main part of traditional CVP analysis, profitability conditions have been examined based optimization method[25] and sphere packing theory[1, 19, 24]. Unlike the previous our work[19, 24], the proposed approach deals with a nonconvex optimization problem using the penalty function method developed in [25]. To find a feasible point from the nonconvex set of profitability conditions, we use the penalty function method and algorithm[25]. Fixing some parameters of a feasible point, we generate sets of profitability conditions with respect to CVP's other parameters. The advantage of the approach in front of traditional CVP analysis is that it can handle multi-product case finding sets of required volumes, prices, and costs which ensure profitability of industries. The sphere packing approach can also analytically represent optimal ranges of parameters via center and radius of a sphere inscribed in a set of profitability conditions. This helps managers to make rational decisions in profitability of a business. The proposed approach was tested on some examples providing numerical results obtained on Matlab.

Declarations

Ethics approval and consent to participate. Not applicable.

Data availability statement. Data created by the authors is accessible to readers.

Competing interests. The authors have no conflicts of interest to declare.

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References

[1] R. Enkhbat. *Recent Advances in Sphere Packing Problems, in Y.Kochetov,A.Eremeev, O.Khamisov, A.Rettieva (Eds.)*, Mathematical Optimization Theory and Operations Research, Springer, 34-35, 2022.

[2] R. Enkhbat. *Convex Maximization Formulation of General Sphere Packing*, The Bulletin of Irkutsk State University, Series Mathematics, 31, 141-142, 2020.

[3] S. R. Abdullahi and et al. *Cost-volume-pro t analysis as a management tool for decision making in small business enterprise within Bayero University*, Kano, IOSR Journal of Business and Management, 19, 40-45, 2017.

[4] S. Ankiti, K. S. Amit, J. Kumar and Sonia. *Profit maximizing hub location problem in the airline industry under coopetition*, Computers and Industrial Engineering, 160, 107-563, 2021.

[5] N. Dopuch and M. W. Maher. *Cost-volume-profit analyses in: R.W. Weil and M.W. Maher(Eds.), Handbook of cost management*, Dunod, Paris, 645-656, 2005.

[6] E. P. Enyi. *Joint Products CVP Analysis Time for Methodical Review*, J. Econ. Bus, 2, 1288-1297, 2019.

[7] Adedeji, Elijah Adeyinka and Ituma, Nneoma Kristen. *Break Even Analysis as A Management Tool for Decision Making in Babcock University Water Corporation*, European Journal of Business and Management, 12, 159-181, 2020.

[8] Lis Sintha Oppusunggu. *Importance of Break-EVEN Analysis for the Micro, Small and Medium Enterprises*, International Journal of Research -GRANTHAALAYAH, 8, 212-218, 2020.

[9] Jurkenaite N and Mikelionyte D. *Raw Milk Price Transmission in the Selected EU Countries*, Scientific Papers Series Management, Economic Engineering in Agriculture and Rural Development, 21, 477-482, 2021.

[10] M. Salam, L. Fudjaja, R. Darma, P. Diansari, N. M. Diansari and M. Nafisa, . *Revenue Analysis of Mobile Rice Milling Business*, IOP Conf. Series: Earth and Environmental Science, 681, 1-5, 2021.

[11] Yan Caoa, Mehrnoush Mohammadian, Vahid Pirouzfar, Chia-Hung Su and Afrasyab Khan. *Break Even Point Analysis of Liquefied Natural Gas Process and Optimization of its Refrigeration Cycles with Technical and Economic Considerations*, Energy, 237, 1461-1477, 2021.

[12] Ilmas Abdurofi, Mohd Mansor Ismail, Normaz Wana Ismail and Amin Mahir Abdullah.

Application of Cost-Benefit and Break Even Analysis for the Development of Stingless Bees Farming in Malaysia, International Journal of Business and Society, 22, 846-861, 2021.

[13] Dario Cottafava, Mattia Costamagna, Marcello Baricco, Laura Corazza, Davide Miceli and Luigi E. Riccardo. *Assessment of the Environmental Break-Even Point for Deposit Return Systems Through an LCA Analysis of Single-Use and Reusable Cups*, Sustainable Production and Consumption, 27, 228-241, 2021.

[14] Zanna Dalatu Gubio, Lateef Olumide Mustapha and Samuel Eniola Agbi. *The Effect of Break-Even-Point Analysis in Decision Making in some selected Block Industries within Kaduna Metropolis*, Quest Journals Journal of Research in Business and Management, 10, 22-32, 2022.

[15] Jan Syrucek, Ludek Barton and Jiri Burdych. *Break-even point Analysis for Milk Production-Selected EU countries*, Agricultural Economics - Czech, 68, 199-206, 2022.

[16] Neidhardt, Michael, Jordi Mas-Peiro, Magnus Schulz-Moenninghoff, Josep O. Pou, Rafael Gonzalez-Olmos, Arno Kwade, and Benedikt Schmuelling. *Forecasting the Global Battery Material Flow: Analyzing the Break-Even Points at Which Secondary Battery Raw Materials Can Substitute Primary Materials in the Battery Production*, Applied Sciences, 12(4), 47-90, 2022.

[17] U. Badam, R. Enkhbat and Ts. Batchimeg.*Application of Survival Theory in a Taxation*, Journal of Indusdrial and Management Optimization, 15(5), 2573-2578, 2021.

[18] R. Enkhbat, N. Tungalag, J. Enkhbayar, O. Battogtokh, and L. Enkhtuvshin, . *Application of Survival Theory in Mining Industry*, Numerical Algebra, Control and Optimization, 11(3), 443-448, 2021.

[19] N. Tungalag and R. Enkhbat. *Application of Sphere Packing Theory in Cost Volume Profit Analysis*, Mongolian Mathematical Journal, 23, 1-6, 2021

[20] Guray Kara, Ayse Ozmen, and Gerhard-Wilhelm Weber *Stability Advances in Robust Portfolio Optimization under Parallelepiped Uncertainty*, CEJOR, 27, 241-261, 2019.

[21] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, 2001.

[22] G. O. Nworie and et al. *Utilizing Cost-Volume-Profit Analysis for Informed Decision Making in Small Business*, Central Asian Journal of Innovations on Tourism Management and Finance, 4(2), 1-14 2023.

[23] Hassan A. Said. *Using Different Probability Distributions for Managerial Accounting Tech-*

nique: The Cost-Volume-Profit Analysis, Journal of Business and Accounting, 9(1), 3-24 2016.

[24] R. Enkhbat and N. Tungalag *A Sphere Packing Approach to Break Even and Profitability Analysis*, Journal of Industrial and Management Optimization, 19(9), 6750-6764, 2023.

[25] Dimitri P. Bertsekas *Nonlinear Programming*, Athena Scientific, 2016.